CHARACTERISTICS OF INFINITE DIMENSIONAL VECTOR SPACES

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ABSTRACT

The study of finite dimensional vector spaces has been logically extended to that of infinite dimensional vector spaces. Of fundamental importance to this study is the relationship between sets which span a vector space, basis sets for such a space, and linearly independent sets within the space. Without recourse to the finite dimensional case, a new proof is presented to show this relationship. A corollary to this is the most important result that every basis for a vector space has the same cardinal number.

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INTRODUCTION

The standard method of proof in showing that all bases for an infinite dimensional vector space have the same cardinal number is first to prove this fact for the finite dimensional case, and then extend it to the infinite case using a set-theoretic argument. [1][2] Using the definition of a basis as a maximal linearly independent set and the theorems developed here that a basis is a linearly independent spanning set and also a minimal spanning set, we shall prove directly the theorem that the cardinal number of any linearly independent set in a vector space is no greater than that of any basis. The proof presented here uses transfinite induction, and is a direct transfinite generalization of the finite-dimensional proofs in [1] and [2]. As a corollary to this theorem we have the essential fact mentioned above that all bases for the same vector space have the same cardinal number.

DEFINITION

A non-empty set V is said to be a vector space over a field F if V is an abelian group under an operation denoted by +, and if for every $\alpha \varepsilon F$, $v \varepsilon V$, there is defined an element in V, written as αv subject to:

(1) $\alpha(v + w) = \alpha v + \alpha w$

(2) $(\alpha + \beta)v = \alpha v + \beta v$

(3) $\alpha(\beta v) = (\alpha\beta)v$

(4) $1v = v$ where 1 is the multiplicative identity in F, i.e., $1\alpha = \alpha$ for all $\alpha \varepsilon F$

Henceforth, we shall call the elements of V vectors, and denote them by lower case Latin letters. We shall call the elements of F scalars, and denote them by lower case Greek letters. Moreover, where no question of ambiguity can arise, we shall omit the words "over the field F", and refer to "the vector space V".

DEFINITION

If V is a vector space over a field F, and if $v_1, \ldots, v_n \varepsilon V$, then any element of the form $\alpha_1 v_1 + \ldots + \alpha_n v_n$, where the α_i are in F, is called a linear combination over F of $v_1, \ldots, v_n$.

The empty sum is, by definition, zero. The reasoning behind this is quite simple, for if $0 < n_1 < n_2$,

$$\sum_{k=0}^{n_2} = \sum_{k=0}^{n_1} + \sum_{k=n_1+1}^{n_2}$$

If equality is to hold also for $n_1 = n_2$, then the last term, which is the empty sum, must be zero.

DEFINITION

If S is any subset of the vector space V, then the Linear span of S, denoted by $L(S)$, is the set of all linear combinations of elements of S. Note that if $\emptyset$ is the empty set, $L(\emptyset) = \{0\}$.

DEFINITION

If V is a vector space over F and if $W \subseteq V$, then W is a subspace of V if, under the operations of V, W itself forms a vector space over F. Equivalently, W is a subspace of V whenever w_1, $w_2 \varepsilon W$, α, $\beta \varepsilon F$ imply that $\alpha w_1 + \beta w_2 \varepsilon W$.

It is easy to see that $L(S)$ is a subspace of V. In fact, $L(S)$ is the smallest subspace of V which contains (always means "$\supseteq$") S, and is equal to the intersection of all subspaces which contain S.

Of particular interest are subsets S of V for which $L(S) = V$. To describe such sets we make the following

DEFINITION

If $L(S) = V$, S is a spanning set, and S is said to span V.

Note that a vector space V always has a spanning set, for at the very worst we always have $L(V) = V$.

DEFINITION

Let V be a vector space over F, and let $D = \{d_1, \ldots, d_n\}$ be a non-empty, finite subset of V. D is linearly dependent if there exist scalars $\lambda_1, \ldots, \lambda_n \varepsilon F$, not all zero, such that $\lambda_1 d_1 + \ldots + \lambda_n d_n = 0$.

DEFINITION

If X is an arbitrary, non-empty subset of V, then X is linearly dependent if X has a finite subset which is linearly dependent. X is linearly independent if it is not linearly dependent.

Note that X is linearly dependent if and only if there exist $x_1, \ldots, x_n \varepsilon X$ and scalars $\lambda_1, \ldots, \lambda_n$, none of which is 0, such that $\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$. Thus we remark that an arbitrary, non-empty set is linearly independent if and only if all of its finite, non-empty subsets are linearly independent. Since the empty set is a finite subset of every set, we are led to make the definition that the empty set, $\emptyset$, is linearly independent.

The following lemma is essential to the work which is to follow.

Lemma 1

Suppose that X is a linearly independent set. Then every element of $L(X)$ can be written in one and only one way as a linear combination of elements of X.

Proof. Case 1: Suppose that $X = \emptyset$. Then $L(X) = \{0\}$, for the only linear combination of elements of X is the empty sum, which is zero.

Case 2: Suppose that $X \neq \emptyset$. By definition of $L(X)$ there is always at least one representation

$$v = \sum_{x \varepsilon X} \lambda_x x$$

where all but finitely many $\lambda_x = 0$. Suppose we have two such representations:

$$v = \sum_{x \varepsilon X} \lambda_x x, \qquad v = \sum_{x \varepsilon X} \mu_x x .$$

Then $0 = v-v = \sum_{x \varepsilon X} (\lambda_x - \mu_x)x$. But X is linearly independent by hypothesis, so that $\sum_{x \varepsilon X} (\lambda_x - \mu_x)x = 0$ implies that $\lambda_x - \mu_x = 0$ for all $x \varepsilon X$. Thus $\lambda_x = \mu_x$ for all $x \varepsilon X$.

DEFINITION

If V is a vector space over F, a subset F of V is a basis for V if B is a maximal linearly independent set. I.e.,

(1) B is linearly independent;

(2) If $B \subset B' \subseteq V$ then B' is linearly dependent.

The next lemma serves to relate the concept of a basis to that of a spanning set.

Lemma 2

B is a basis for the vector space V if and only if B is simultaneously

(1) a linearly independent set, and

(2) a spanning set for V.

Proof. Suppose that B is a basis, but that B is not a spanning set. Then $L(B) \subset V$, so we may choose $y \varepsilon V - L(B)$. Let $B' = B \cup \{y\} \supset B$. Suppose B' is linearly dependent. Then there exist $b_1, \ldots, b_n \varepsilon B'$ and non-zero scalars $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 b_1 + \ldots + \lambda_n b_n = 0$. Now one of the b_i must be y, for otherwise B would be dependent; we may assume $b_1 = y$. Then $\lambda_1 y = -\lambda_2 b_2 - \ldots - \lambda_n b_n$; hence, $y = (-\lambda_2/\lambda_1)b_2 + \ldots + (-\lambda_n/\lambda_1)b_n$,

which implies that $y \varepsilon L(B)$. This is a contradiction so that B' is linearly independent. This contradicts the hypothesis that, as a basis, B is a maximal linearly independent set. Hence, B spans V.

Conversely, suppose B is a linearly independent set which spans V, but that B is not a basis for V. Then there exists a linearly independent set B' such that $B \subset B'$. Choose $y \varepsilon B' - B$ and consider $S = B \cup \{y\} \subseteq B'$. S is linearly independent, since it is a subset of B', so that we cannot express y as a linear combination of the other elements of S. This implies $y \notin L(B)$, contrary to hypothesis, since $y \varepsilon B' - B \subset V$ and $L(B) = V$. Hence, B is a basis for V.

The next lemma will describe a basis for us completely in terms of spanning sets.

Lemma 3

B is a basis for a vector space V if and only if B is a minimal spanning set. I.e.,

(1) B is a spanning set, and

(2) If $B' \subset B$, then B' does not span V.

Proof. If B is a basis for V, then B is a maximal linearly independent set, and moreover B spans V. Suppose that there exists a spanning set $B' \subset B$. Then B' is linearly independent since it is a subset of a linearly independent set. By Lemma 2, B' is a basis, and hence a maximal linearly independent subset which is a proper subset of the linearly independent set B. This is a contradiction.

Conversely, suppose B is a minimal spanning set, but that B is not a basis for V. By Lemma 2, B is then linearly dependent, so that there exist $b_1, \ldots, b_n \varepsilon B$ and non-zero scalars $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 b_1 + \ldots + \lambda_n b_n = 0$. Since $\lambda_1 \neq 0$, $b_1 = (-\lambda_2/\lambda_1)b_2 + \ldots + (-\lambda_n/\lambda_1)b_n$. Thus any linear combination of elements of B which involves B_1 can be written as a linear combination of elements of B without using b_1. Therefore, $B' = B - \{b_1\}$ is a spanning set for V such that $B' \subset B$. This contradicts the fact that B is a minimal spanning set. Hence, B is linearly independent and, as a spanning set, by Lemma 2, is a basis.

Examples

(1) Let E^n be the set of all ordered n-tuples with real components. Then the vectors $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, ..., $e_n = (0,0,0,\ldots,1)$ form a basis for E^n.

(2) The set $Q^{\aleph_0}$ of all polynomials with rational coefficients forms a vector space, and the set $\{1,x,x^2,x^3,\ldots\}$ is a basis for this space.

(3) The real numbers may be considered as a vector space over the field of rational numbers. However, no one has ever been able to exhibit a basis for this space (although, as we shall show later, one does exist).

We have so far defined a vector space and discussed a certain kind of subset of it--namely, a basis. The question which one should now investigate is the general existence of such a subset. That is, does every vector space have a basis?

Before specifically investigating this fact, we develop certain preliminary results. Since we intend to deal ultimately with infinite sets, we introduce the Axiom of Choice.

Axiom of Choice

Let $\mathcal{R}$ be any non-empty family of non-empty sets. Then there exists a set X such that for every set F in the family $\mathcal{R}$, $X \cap F$ is a singleton. I.e., we can "choose" one element from each set F in the family.

There are many equivalent forms of the Axiom of Choice.

DEFINITION

A well-ordered set is a partially-ordered set in which every non-empty subset S contains a minimal element, i.e., an element x such that $x \leq y$ for every $y \in S$.

DEFINITION

A subset S of a partially ordered set is called a chain if for every $x, y \in S$ either $x \leq y$ or $y \leq x$.

With these definitions, we are now able to state several of the equivalent forms of the Axiom of Choice.

Zermelo's Theorem [3]

Every set can be well-ordered.

Examples:

The natural numbers are well-ordered by their natural ordering; the real numbers are not. However, Zermelo's theorem says that the real numbers can be well-ordered, but no one has done so constructively.

Zorn's Lemma [4]

If S is a partially ordered set such that every chain in S has an upper bound, then S contains a maximal element.

Theorem 1

Let L be a linearly independent subset of V, and let $S \supseteq L$ be a spanning subset of V. Then there exists a basis B of V such that $S \supseteq B \supseteq L$.

Proof. The class $\mathfrak{K}$ of all linearly independent subsets of S can be partially ordered by the relation of set inclusion. Since $L \varepsilon \mathfrak{K}$, $\mathfrak{K}$ is not empty. Let $\mathcal{K} = \{A_\alpha\}$ be a chain in $\mathfrak{K}$, and let P be the union of the sets in $\mathcal{K}$. Obviously, $P \subseteq S$. We claim that P is linearly independent so that $P \varepsilon \mathfrak{K}$. Suppose that P were linearly dependent. Then it must contain a finite linearly dependent subset $P' = \{p_1, p_2, \ldots, p_n\}$. Since each p_i is an element of some $A\alpha_i$, and since the chain is simply ordered, one of the $A\alpha_i$ must contain all of the rest, so that it must contain P'. This implies that this particular $A\alpha_i$ is linearly dependent, which is a contradiction. Therefore, P is linearly independent and clearly serves as an upper bound for the elements in $\mathcal{K}$. Thus $\mathfrak{K}$ satisfies the conditions of Zorn's lemma, so it contains a maximal (i.e., maximal in S) linearly independent set which we shall call B. But B spans V (i.e., is maximal in V also), for suppose there exists an element $y \varepsilon V - L(B) = L(S) - L(B)$. Then $y = \lambda_1 s_1 + \ldots + \lambda_n s_n$, where the $s_i \varepsilon S$ and at least one s_i, say s_1,

is not in $L(B)$. Then $B \cup \{s_1\}$ is a linearly independent set in S which properly contains B, contradicting the maximality of B. Hence, $L(B) = L(S) = V$, and by Lemma 2, B is a basis for V.

Since V spans V, we have the following corollary:

<u>Corollary 1</u>

If L is any linearly independent subset of V, then L is contained in a maximal linearly independent subset, i.e., in a basis.

Since $\emptyset$ is a linearly independent subset of V, we have

<u>Corollary 2</u>

If S is any spanning subset of V, then S contains a minimal spanning subset, i.e., a basis.

The most important corollary is the following:

<u>Corollary 3</u>

Every vector space has a basis.

We now pause for a brief recapitulation. We have under consideration a structure called a vector space, denoted by V. We have developed various properties of certain subsets of V, namely, linearly independent subsets, spanning sets, and bases. Since bases are maximal linearly independent sets and minimal spanning sets, in some sense, bases should be "smaller" than spanning sets, but "larger" than linearly independent

sets. In Theorem 2, we shall see that this is true in a precise sense, namely, that if L is any linearly independent set, S any spanning set, and B any basis, then $\#(L) \leq \#(B) \leq \#(S)$, where $\#(X)$ is the cardinal number of the set X.

DEFINITION

If S and T are subspaces of the vector space V, then V is said to be the direct sum of S and T, denoted $V = S \oplus T$, if every $v \in V$ can be written uniquely as $v = s + t$ where $s \in S$ and $t \in T$.

Lemma 4

Suppose that S is a subspace of the vector space V. Then there exists a subspace T of V such that $V = S \oplus T$. T is called the complement of S.

Proof. Let B be a basis for S and extend B to a basis D for V which is possible by Theorem 1. Let $T = L(D-B)$. It is easy to see that $V = S \oplus T$.

The standard method of proof for Theorem 2, as given in [1] and [2], is first to prove the result for the finite-dimensional case, and then to extend to the infinite-dimensional case using a set-theoretic cardinality argument. Our proof is a direct transfinite generalization of the finite-dimensional argument; it works equally well for either finite or infinite-dimensional spaces. Raikov [6] also has a direct proof of the theorem.

Theorem 2

Suppose that X is a linearly independent subset of the vector space V and that Y is a basis for V. Then $\#(X) \leq \#(Y)$.

Proof. Assume without loss of generality that X and Y are disjoint. For if they are not, let $S = X \cap Y$ and let T be the complement of S in V. Let $X' = X - S$ and $Y' = Y - S$. Then Y' is a basis for T, and X' is a linearly independent subset of T. If we can show that $\#(X') \leq \#(Y')$ it will follow that $\#(X) \leq \#(Y)$.

We now assume that X and Y are disjoint. Well-order Y, using Zermelo's theorem:

$$Y = \{y_0, y_1, \ldots, y_\omega, y_{\omega+1}, \ldots\} .$$

Let the ordinal number of this set be γ so that y_α is defined for all ordinals $\alpha < \gamma$, but not for $\alpha = \gamma$. We shall define by transfinite induction a transfinite sequence $x_0, x_1, \ldots, x_\omega, x_{\omega+1}, \ldots$ of distinct elements of X which will exhaust X. The ordinal number of this sequence will be $\leq \gamma$ so that it will follow that $\#(X) \leq \#(Y)$.

Suppose that x_β has been defined for all $\beta < \alpha < \gamma$. Let

$$X_\alpha = \bigcup_{\beta<\alpha} \{x_\beta\}$$

$$Y_\alpha = \bigcup_{\beta<\alpha} \{y_\beta\}$$

$$W_\alpha = X \cup Y_\alpha - X_\alpha$$

define X_β, Y_β, and W_β in the obvious fashion for $\beta<\alpha$ so that $W_0 = X$. It will be shown that W_α is always linearly independent. Suppose further

that W_β is independent for $\beta<\alpha$. (Note that $W_0 = X$ is independent by assumption.) Assume for the moment that W_α has been shown to be independent. We shall show how to select x_α so that $W_{\alpha+1}$ is independent. A corollary of this construction is that if α is a successor ordinal, W_α is independent (just back up one step). We shall show separately later that W_α is independent if α is a limit ordinal.

If $X \cap W_\alpha = X - X_\alpha = \emptyset$, we are through, since $X = X_\alpha$ and $\alpha<\gamma$, so suppose that $X \cap W_\alpha \neq \emptyset$. Consider now $y_\alpha \varepsilon Y - W_\alpha = Y - Y_\alpha$. We distinguish between two cases:

<u>Case 1</u>: $y_\alpha \notin L(W_\alpha)$. In this case select x_α arbitrarily in $X \cap W_\alpha = X - X_\alpha$. Since $y_\alpha \notin L(W_\alpha)$, it follows that $W_\alpha \cup \{y_\alpha\}$ is independent, and hence that $W_{\alpha+1} = W \cup \{y_\alpha\} - \{x_\alpha\}$ is also independent.

<u>Case 2</u>: $y_\alpha \varepsilon L(W_\alpha)$. In this case we can write y_α uniquely as a linear combination of elements of W_α as follows:

$$y_\alpha = \lambda_1 w_1 + \lambda_2 w_2 + \ldots + \lambda_n w_n \ . \qquad (*)$$

where all $\lambda_i \neq 0$ and where each $w_i \varepsilon W_\alpha$. Now Y is a basis, so that $y_\alpha \notin L(Y_\alpha) \equiv L(Y \cap W_\alpha)$. This implies that at least one $w_i \varepsilon X \cap W_\alpha$. Assume without loss of generality that $w_1 \varepsilon X \cap W_\alpha$. Now let $x_\alpha = w_1$. Then $W_{\alpha+1} = W_\alpha \cup \{y_\alpha\} - \{x_\alpha\}$ is independent, for suppose, to the contrary, that it is not. Then there exist non-zero scalars $\mu_0, \mu_1, \ldots, \mu_m$ and elements $w'_1, w'_2, \ldots, w'_m \ \varepsilon \ W_\alpha - \{x_\alpha\}$ such that

$$\mu_0 y_\alpha + \mu_1 w'_1 + \mu_2 w'_2 + \ldots + \mu_m w'_m = 0 \ ,$$

where we know that $\mu_0 \neq 0$ since W_α is independent. This implies that y_α can be written

$$y_\alpha = (-\mu_1/\mu_0)w_1' + \ldots + (-\mu_m/\mu_0)w_m'$$

where no w_i' is equal to $w_1 = x_\alpha$. This contradicts the uniqueness of the representation (*) of y_α. Therefore, $W_{\alpha+1}$ is independent.

Now we shall show that W_α must have been independent to start with. In out construction of x_α, we showed that $W_{\alpha+1}$ was independent whenever W_α was independent. Since $W_0 = X$ is independent by assumption, this proves that if α is a successor ordinal, then W_α must have been independent, for all $\alpha \leq \gamma$. Suppose then that $\alpha \leq \gamma$ is a limit ordinal, and suppose that W_α were dependent. Then there exist $w_1, w_2, \ldots, w_n \varepsilon W_\alpha$ and non-zero scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1 w_1 + \ldots + \lambda_n w_n = 0$. Since

$$W_\alpha = (\bigcap_{\beta<\alpha} X \cap W_\beta) \cup (\bigcup_{\beta<\alpha} Y \cap W_\beta),$$

it follows that $w_i \varepsilon W_{\beta_i}$ for some $\beta_i < \alpha$. Let β_0 be the largest index in the set of indices $\{\beta_1, \ldots, \beta_n\}$. Then for all i, $w_i \varepsilon W_{\beta_0}$. Since $\beta_0 < \alpha$, W_{β_0} is linearly independent by assumption; this is a contradiction.

There are two possibilities. The first is that at some ordinal $\alpha<\gamma$, $X - X_\alpha = \emptyset$ so that $X = X_\alpha$ and the chain will stop. In this case, we are through, since $X = X_\alpha$ and $Y = Y_\gamma \supseteq Y_\alpha$ imply that $\#(X) \leq \#(Y)$. The other possibility is that we can define x_α for all $\alpha<\gamma$. Then $W_\gamma = X \cup Y_\gamma - X_\gamma$ is independent. But $Y_\gamma = Y$ and $W_\gamma \supseteq Y$ so that $W_\gamma = Y$ since Y is a basis. However, X and Y are disjoint so that $X - X_\gamma = \emptyset$, i.e., $X = X_\gamma$. This implies that $\#(X) = \#(X_\gamma) = \#(Y_\gamma) = \#(Y)$, and the theorem is proved.

The theorem just proved shows that if L is a linearly independent set of vectors in a vector space V and B is a basis for V, then $\#(L) \leq \#(B)$. From this theorem we have the following important corollaries:

Corollary 1

If B_1 and B_2 are two bases for V, then B_1 is equivalent to B_2. I.e., $\#(B_1) = \#(B_2)$.

Proof. As bases, B_1 and B_2 are both linearly independent sets. Considering B_1 as linearly independent and B_2 as a basis, we have from the theorem that $\#(B_1) \leq \#(B_2)$. Reversing the roles, we have that $\#(B_2) \leq \#(B_1)$. Thus by the Schröder-Bernstein theorem [5] on cardinal numbers, $\#(B_1) = \#(B_2)$.

Corollary 2

Let B be a basis for V, let L be a linearly independent set in V, and let S be a spanning set of V. Then $\#(L) \leq \#(B) \leq \#(S)$.

Proof. By Theorem 1, the spanning set S contains a basis B', so that $\#(S) \geq \#(B')$. By Corollary 1, $\#(B') = \#(B)$. Putting this together with Theorem 2 yields $\#(L) \leq \#(B) \leq \#(S)$.

These are most important results for they show that, whereas a basis for a vector space may not be unique, any two bases will have the same cardinality.

DEFINITION

The cardinality of any basis of a vector space V is called the dimension of V. This is unambiguous by the previous theorem.

We can now generalize the definition of direct sum from two to an arbitrary number of summands.

DEFINITION

Let $\{A_\alpha\}_{\alpha\varepsilon\Lambda}$ be a family of vector spaces over the same field F. Then the direct sum of the spaces A_α, denoted $\oplus \sum_{\alpha\varepsilon\Lambda} A_\alpha$, is the set of all functions f which map the index set Λ into the union of the A_α, with the restrictions that $f(\alpha)\varepsilon A_\alpha$ and that $f(\alpha)$ must be 0 except for finitely many indices. Intuitively, the direct sum is the set of all "α-tuples" with only finitely many non-zero coordinates. The usual functional operations on the direct sum make it into a vector space over F. A case of particular interest is when each A_α is the field F itself. Let μ be a cardinal number, and let Λ be an index set of cardinality μ. Then by F^μ we mean the direct sum $\oplus \sum_{\alpha\varepsilon\Lambda} A_\alpha$, where for every α, $A_\alpha = F$. It is easy to see that the direct sum is essentially independent of which index set is chosen, as long as it has cardinality μ. F^μ is intuitively a direct sum of "μ copies" of F. Note that F^μ is the set of all functions f from a μ-element index set Λ to F where $f(\alpha) = 0$ except for finitely many values of α.

Lemma 5

F^μ is a vector space over F and the dimension of F^μ is μ.

Proof. Let Λ be a μ-element index set. For every $\alpha\varepsilon\Lambda$, define $e_\alpha \varepsilon F^\mu$ as follows:

$$e_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}$$

It is obvious that the set $B = \{e_\alpha : \alpha\varepsilon\Lambda\}$ is a basis for F^μ, so that the dimension of $F^\mu = \#(B) = \mu$.

Lemma 6

If V is any vector space over F of dimension μ, then V is isomorphic to F^μ.

Proof. Suppose $B' = \{x_\alpha\}_{\alpha\varepsilon\Lambda}$ is a basis for V, where $\#(\Lambda) = \mu$. We have shown (Lemma 5) that $B = \{e_\alpha : \alpha\varepsilon\Lambda\}$ is a basis for F^μ. Define $\phi : V \to F^\mu$ as follows: If $v\varepsilon V$, write v uniquely as $v = \lambda_1 x_{\alpha_1} + \ldots + \lambda_n x_{\alpha_n}$ where all $\lambda_i \neq 0$ and where each $x_{\alpha_i} \varepsilon B'$. Then $v\phi$ is the function εF^μ defined as follows:

$$v\phi(\alpha) = \begin{cases} \lambda_i \text{ if } \alpha = \alpha_i \text{ for some } i \\ 0 \text{ if } \alpha \neq \alpha_i \text{ for all } i. \end{cases}$$

It is a routine computation to show that ϕ is an isomorphism of V onto F^μ and moreover that $x_\alpha\phi = e_\alpha$ for every $\alpha\varepsilon\Lambda$.

Lemma 7

Suppose that $\phi : V \to W$ is an isomorphism of V onto W. If $L \subseteq V$ is linearly independent then $\phi(L) \subseteq W$ is linearly independent, and if $S \subseteq V$ is a spanning set then $\phi(S) \subseteq W$ is a spanning set. Consequently, if $B \subseteq V$ is a basis, $\phi(B) \subseteq W$ is a basis.

Proof. Suppose that $L \subseteq V$ is linearly independent and that $\lambda_1 w_1 + \ldots + \lambda_n w_n = 0$ where each $w_i \varepsilon \phi(L)$. Let $v_i = \phi^{-1}(w_i)$ so that $v_i \varepsilon L$. Then

$\lambda_1 v_1 + \ldots + \lambda_n v_n = \lambda_1 \phi^{-1}(w_1) + \ldots + \lambda_n \phi^{-1}(w_n) = \phi^{-1}(\lambda_1 w_1 + \ldots + \lambda_n w_n) =$ $\phi^{-1}(0) = 0$, since ϕ^{-1} is an isomorphism. Since L is linearly independent,

$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$ so that $\phi(L)$ is linearly independent.

Suppose now that $S \subseteq V$ is a spanning set. Let $w \varepsilon W$ be arbitrary and consider $v = \phi^{-1}(w) \varepsilon V$. Now $v \varepsilon L(S)$ so that $v = \lambda_1 s_1 + \ldots + \lambda_n s_n$ for certain scalars λ_i and $s_i \varepsilon S$. But $w = \phi(v) = \phi(\lambda_1 s_1 + \ldots + \lambda_n s_n) = \lambda_1 \phi(s_1) + \ldots + \lambda_n \phi(s_n)$ which is a linear combination of elements of $\phi(S)$. That is, $w \varepsilon L(\phi(S))$. Since w was arbitrary, $\phi(S)$ spans W.

The following corollary is quite important.

Corollary

If V is isomorphic to W, then V and W have the same dimension.

We are now in a position to characterize completely all vector spaces over an arbitrary field F.

Theorem 3

A vector space V over a field F is completely characterized by its dimension. I.e., if V and W are vector spaces over the same field F, then V is isomorphic to W if and only if the dimension of V is equal to the dimension of W. Moreover, if the dimension of V is μ, then V is isomorphic to F^{μ}.

Proof. If V is isomorphic to W, then by the corollary to Lemma 7, V and W have the same dimension. Conversely, if V and W both have dimension μ, we have by Lemma 6 that both V and W are isomorphic to F^{μ}, and hence isomorphic to each other.

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13. ABSTRACT

The study of finite dimensional vector spaces has been logically extended to that of infinite dimensional vector spaces. Of fundamental importance to this study is the relationship between sets which span a vector space, basis sets for such a space, and linearly independent sets within the space. Without recourse to the finite dimensional case, a new proof is presented to show this relationship. A corollary to this is the most important result that every basis for a vector space has the same cardinal number.

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